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LANDAU DAMPING AND GROWTH OF ELECTROSTATIC

MODES WITH FINITE SYSTEM EFFECTS

by

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ABSTRACT

For a plasma with finite cross-section of the sort which occurs typically in laboratory plasma waves experiments, in a constant magnetic field, the Landau damping (or growth) is obtained by a variational procedure in terms of the plasma velocity distribution function and potential and density profiles. The result is applied to the damping associated with the upper and lower branches of the dispersion curve for longitudinal electron plasma waves in the case of a Maxwellian velocity distribution in slab and cylindrical geometries. Application is also made to the growth rates resulting from a low density electron beam with radially dependent energy superposed on a Maxwellian.

I. INTRODUCTION

Theoretical investigations of the dispersion relations and damping of longitudinal electron plasma waves[1] have been carried out in considerable detail for the case of homogeneous finite-temperature collisionless plasma. However, it is of interest to extend these considerations to inhomogeneous systems as well, since laboratory experiments[2] always employ bounded plasmas.

A number of other papers have treated this subject. Trivelpiece and Gould[3] considered a cylindrical cold plasma inside a concentric cylindrical conducting surface in the quasi-static (low β) approximation. The boundary conditions lead to a dispersion relation with two branches, a lower one corresponding to Langmuir oscillation, and an upper one, the "backward wave", near the cyclotron frequency. Gould[4] generalized this result for the lower branch in the case of strong magnetic field ($\Omega_e \gg \omega_{pe}$), using the finite-temperature dielectric tensor and treating a smoothly varying cross-sectional density profile instead of a step function. The effect of finite temperature is to introduce Landau damping and to permit ω to become greater than ω_{pe} for large $k_{\parallel}$, just as in the homogeneous case.

Lichtenberg and Jayson[5] consider one- and two-stream Maxwellian plasmas, keeping lowest order finite temperature terms in the dielectric tensor. For a cylindrical plasma with constant density bounded at $r = R$ (step-function dependence), they solve the dispersion relation for the decay (growth) rate in terms of frequency, wave length and plasma density, temperature and radius. The contribution to this growth rate comes from resonance between electrons and three plasma modes: the Langmuir oscillations, the $n = 1$ cyclotron mode (backward wave) and the $n = -1$ mode.

In this paper an approach similar to that of Lichtenberg and Jayson is employed. In Section II a variational technique is used to calculate the decay rate γ as a small perturbation correction to the wave frequency. Here a Maxwellian electron distribution is assumed, but the density and electric potential are arbitrary slowly varying functions of position, regarded as being obtained from measurements. They enter in the final expression for γ only in two rather insensitive averages. The result describes damping in both upper and lower branches. Formulas for the growth rate in both 2-dimensional (slab) and cylindrical geometry are derived, for strong $(\Omega_e > \omega_{pe})$ and weak $(\Omega_e < \omega_{pe})$ magnetic field.

In Section III, this calculation is modified to include the effect of a low density electron beam. The beam is injected axially with a very narrow velocity spread about a mean velocity which is a function of position. This injected beam is shown to lead to an instability of the lower branch mode which may be made quite gentle, so as to be describable by the quasi-linear theory[6].

II. DECAY RATE FOR MAXWELLIAN PLASMA

We will consider a plasma which is uniform in the direction along the uniform constant magnetic field (the z-direction) and inhomogeneous in the transverse direction. We restrict ourselves to low β (quasi-static) systems, so that the perturbed electric field is derivable from a potential. From the linearized Vlasov equation we can obtain Poisson's equation in the form

$$\nabla \cdot \underset{\approx}{\epsilon} \cdot \nabla\varphi = 0. \tag{1}$$

We discuss systems with two-dimensional geometry first, then outline the analogous treatment of cylindrical systems.

For two-dimensional geometry we write the potential as

$$\varphi = \psi(x)e^{ik_{\parallel}z-i\omega t}. \tag{2}$$

$\psi(x)$ varies smoothly over the cross section of the plasma, so we can speak of an effective transverse wave number $k_{\perp} \sim 1/R$, where R is the plasma width or radius. We assume Maxwellian velocity distributions with ion temperatures not substantially larger than electron temperatures, so only the electron contribution to $\underset{\approx}{\epsilon}$ need be retained. Then for long wavelengths

$$\frac{\omega}{k_{\parallel}} \gg v_T, \tag{3}$$

and small electron Larmor radius

$$\frac{\rho_e}{R} \ll 1, \tag{4}$$

$\underset{\approx}{\epsilon}$ is given by[7]

$$\underset{\approx}{\epsilon} = \hat{x}\hat{x}\epsilon_{\perp} + \hat{z}\hat{z}\epsilon_{\parallel} \tag{5}$$

where

$$\epsilon_{\parallel} = 1 + \sum_{n=-\infty}^{\infty} \frac{2\omega_p^2}{\omega k_{\parallel} v_T} e^{-z} I_n(z) x_n [1 + x_n Z(x_n)] \tag{6}$$

and

$$\epsilon_{\perp} = 1 + \sum_{n=-\infty}^{\infty} \frac{\omega_p^2}{\omega k_{\parallel} v_T} Z(x_n) e^{-z} \frac{n^2}{z} I_n(z). \tag{7}$$

Here ω_p is the electron plasma frequency, $\omega_p^2 = \frac{4\pi e^2}{m} n(x)$; Ω_e is the electron cyclotron frequency; $v_T^2 = \frac{2T}{m}$;

$$z = \frac{k_{\perp}^2 T}{\Omega_e^2 m} \ll 1, \qquad \text{by (4);}$$

$$x_n = \frac{\omega - n\Omega_e}{k_{\parallel} v_T} \gg 1 \text{ (unless } \omega \approx n\Omega_e\text{), by (3);}$$

Z is the plasma dispersion function of Fried and Conte[8], defined by

$$Z(\zeta) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx \, \frac{e^{-x^2}}{x-\zeta} \tag{8}$$

$$\longrightarrow i\pi^{\frac{1}{2}} e^{-\zeta^2} - \frac{1}{\zeta}\left[1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^2} + \ldots\right]$$

asymptotically for large argument; and I_n is the Bessel function of imaginary argument of n^{th} order

$$I_n(z) \sim (\tfrac{1}{2}z)^n \left[1 + 0(z^2)\right]$$

for small z.

In Eq. (5) diagonal terms and corrections to $\epsilon_{\parallel}$ and $\epsilon_{\perp}$ arising from the variation of the density n have been dropped by Eq. (3) and Eq. (4); inclusion of the latter can lead in some cases to the existence of unstable drift modes, the universal instability.

Next we expand Eq. (6) and Eq. (7) for large x_n and small z, using the asymptotic form Eq. (8). If in addition we write $\omega = \omega_0 + i\gamma$, $\gamma/\omega_0 \ll 1$, and retain only terms to first order in γ/ω_0 and $\exp(-x_n^2)$, the result is

$$\epsilon_{\parallel} = 1 - \frac{\omega_p^2}{\omega_0^2} + i\left\{2\pi^{\frac{1}{2}}\omega_p^2 \frac{\omega_0}{(\omega_{\parallel} v_T)^3} e^{-\left(\frac{\omega}{k_{\parallel} v_T}\right)^2} + \frac{\gamma\,\omega_p^2}{\frac{4}{\omega_0}}\right\} \tag{9}$$

$$\epsilon_{\perp} = 1 - \frac{\omega_p^2}{\omega_0^2-\Omega_e^2} + i\left\{\frac{1}{2}\pi^{\frac{1}{2}}\frac{\omega_p^2}{\omega_0 k_{\parallel} v_T}\left[\exp - \left(\frac{\omega_0-\Omega_e}{k_{\parallel} v_T}\right)^2 + \exp - \left(\frac{\omega_0+\Omega_e}{k_{\parallel} v_T}\right)^2\right]\right.$$

$$\left. + \frac{2\gamma\omega_0\omega_p^2}{(\omega_0^2-\Omega_e^2)^2}\right\} \tag{10}$$

In Eq. (10), contributions from higher harmonics ($|n| > 1$) are exponentially small unless $n\Omega_e \simeq \omega$; then they are small like some power of z. Unless the magnetic field is very weak, it is usually possible to neglect the n = -1 term as well, and this will be done in what follows.

Substitution of Eq. (5) in Eq. (1) yields

$$\frac{d}{dx}\left(\epsilon_\perp \frac{\partial\psi}{\partial x}\right) - k_\parallel^2 \epsilon_\parallel \psi = 0. \tag{11}$$

For given boundary conditions and density profile $n(x)$, this equation is an eigenvalue problem which in principle yields the analytic form of ψ and a dispersion relation for complex ω as a function of $k_\parallel$. In general it is not possible to carry out this calculation exactly, even in the limit $T \rightarrow 0$ where ω is real and the modes propagate undamped.

Nevertheless it is possible to utilize Eq. (11) in a variational approach. To do this, we rewrite it in the form

$$L\psi = (L_0 + iL_1)\psi = 0 \tag{12}$$

Here L_0 and L_1 are the real and imaginary parts of L:

$$L_j = \frac{d}{dx}\,\epsilon_\perp^{(j)}\,\frac{d}{dx} - k_\parallel^2 \epsilon_\parallel^{(j)}, \qquad j = 0,\ 1;$$

$$\epsilon_\perp^{(0)} = 1 - \frac{\omega_p^2}{\omega_0^2-\Omega_e^2}, \qquad \epsilon_\parallel^{(0)} = 1 - \frac{\omega_p^2}{\omega_0^2};$$

$$\epsilon_\perp^{(1)} = \frac{2\gamma\omega_0\omega_p^2}{(\omega_0^2-\Omega_e^2)^2} + \frac{1}{2}\pi^{\frac{1}{2}}\frac{\omega_p^2}{\omega_0 k_\parallel v_T}\exp - \left(\frac{\omega_0-\Omega_e}{k_\parallel v_T}\right)^2 ;$$

$$\epsilon_\parallel^{(1)} = \frac{\gamma\omega_p^2}{\omega_0^4} + 2\pi^{\frac{1}{2}}\,\omega_p^2\,\frac{\omega_0}{(k_\parallel v_T)^3}\exp - \left(\frac{\omega}{k_\parallel v_T}\right)^2 .$$

Now consider the equation

$$L_0\psi_0 = 0 \tag{13}$$

which describes undamped waves propagating at T = 0. Multiply Eq. (12) by $\psi_0^*(x)$, Eq. (13) by $\psi^*(x)$, and integrate over x from $-\infty$ to $+\infty$:

$$\int_{-\infty}^{\infty} dx\ \psi_0^* L_0 \psi + i \int_{-\infty}^{\infty} dx\ \psi_0^* L_1 \psi = 0 \tag{14}$$

$$\int_{-\infty}^{\infty} dx\ \psi^* L_0 \psi_0 = \int_{-\infty}^{\infty} dx\ \psi_0^* L_0 \psi = 0 \tag{15}$$

since L_0 is self-adjoint. Finally, subtracting Eq. (15) from Eq. (14), approximating $\psi_0 \approx \psi$ and integrating by parts yields

$$\int_{-\infty}^{\infty} dx\ \frac{2\omega_p^2 k_\parallel^2 \gamma}{\omega_0^3} \left|\psi(x)\right|^2 + \int_{-\infty}^{\infty} dx\ \frac{2\omega_p^2 \gamma \omega_0}{(\omega_0^2 - \Omega_e^2)^2} \left|\frac{d\psi}{dx}\right|^2$$

$$= - \int_{-\infty}^{\infty} dx\ \frac{\frac{1}{2}\pi^{1/2}\omega_p^2}{\omega_0 k_\parallel v_T} \exp - \left(\frac{\omega_0 - \Omega_e}{k_\parallel v_T}\right)^2 \left|\frac{d\psi}{dx}\right|^2 \tag{16}$$

$$- \int_{-\infty}^{\infty} dx\ \frac{2\pi^{1/2} k_\parallel^2 \omega_p^2 \omega_0}{(k_\parallel v_T)^3} \exp - \left(\frac{\omega_0}{k_\parallel v_T}\right)^2 \left|\psi(x)\right|^2 .$$

This may be solved for γ as

$$\frac{\gamma}{\omega_0} = - \frac{\pi^{\frac{1}{2}} \frac{\omega_0^3}{(k_{\parallel} v_T)^3} \exp - \left(\frac{\omega_0}{k_{\parallel} v_T}\right)^2 G + \frac{\pi^{\frac{1}{2}}}{4} \frac{\omega_0}{k_{\parallel} v_T} \exp - \left(\frac{\omega_0 - \Omega_e}{k_{\parallel} v_T}\right)^2 F}{G + \frac{\omega_0^4}{(\omega_0^2 - \Omega_e^2)^2} F} \tag{17}$$

where

$$F = \int_{-\infty}^{\infty} dx\, \omega_p^2 \left|\frac{d\psi}{dx}\right|^2 \tag{18}$$

$$G = k_{\parallel}^2 \int_{-\infty}^{\infty} dx\, \omega_p^2 \left|\psi(x)\right|^2 \tag{19}$$

For an approximately homogeneous plasma in which $\Omega_e \gg \omega_p$, the dispersion curve looks as shown in Fig. 1. On the lower branch, $\omega_0 \ll \Omega_e$, and the equation for γ reduces to

$$\frac{\gamma}{\omega_0} = - \frac{\pi^{\frac{1}{2}} \frac{\omega_0^3}{(k_{\parallel} v_T)^3} e^{-\left(\frac{\omega_0}{k_{\parallel} v_T}\right)^2} G}{\left(\frac{\omega_0}{\Omega_e}\right)^4 F + G} \tag{20}$$

The calculations of Trivelpiece and Gould[3] and of Gould[4] show that for small $k_{\parallel}$, ω_0/Ω_e goes like $k_{\parallel}R$. Since $F/G \sim (k_{\parallel}R)^{-2}$, we see that Eq. (20) reduces to the familiar expression for the Landau damping of a homogeneous plasma in this limit, as it also does in the limit $R \to \infty$. On the upper branch, $\omega_0 \sim \Omega_e$, and

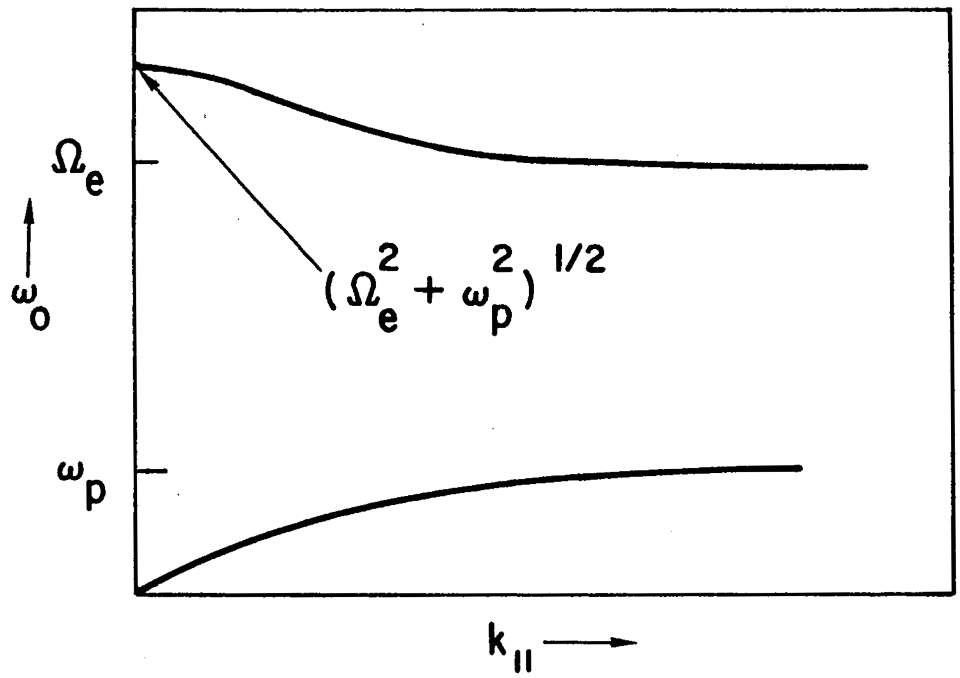

Fig. 1 -- Dispersion curve for homogeneous plasma in strong field.

$$\frac{\gamma}{\omega_0} = - \frac{\frac{1}{4}\pi^{\frac{1}{2}} \frac{\omega_0}{k_\parallel v_T} e^{-\left(\frac{\omega_0-\Omega_e}{k_\parallel v_T}\right)^2} F}{\frac{\omega_0^4}{(\omega_0^2-\Omega_e^2)^2} F + G}, \tag{21}$$

which is valid provided $|\omega_0-\Omega_e|$ is not small compared with $k_\parallel v_T$.

If $\omega_p \gg \Omega_e$ through the bulk of the plasma, there are still two branches; on the lower one, $\omega_0 \lesssim \Omega_e$, and on the upper one, $\omega_0 \gg \Omega_e$. Now Eq. (17) does not simplify.

For the case of cylindrical symmetry we assume the potential is azimuthally symmetric:

$$\varphi = \psi(r)e^{ik_\parallel z-i\omega t}, \tag{22}$$

and Eq. (11) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\epsilon_\perp \frac{d\psi}{dr}\right) - k_\parallel^2\epsilon_\parallel\psi = 0. \tag{23}$$

Here $\epsilon_\parallel$ and $\epsilon_\perp$ are again given by Eq. (6) and Eq. (7), since in three dimension $\underset{\approx}{\epsilon}$ is diagonal and $\epsilon_{yy} \approx \epsilon_{xx} \approx \epsilon_\perp$ in the approximations Eq. (3) and Eq. (4).

The variational calculation goes through unchanged, except that now

$$L_0 = \frac{d}{dr}\left(r\epsilon_\perp^{(0)} \frac{d}{dr}\right) - k_\parallel^2\epsilon_\parallel^{(0)};$$

$$L_1 = \frac{d}{dr}\left(r\epsilon_\perp^{(1)} \frac{d}{dr}\right) - k_\parallel^2\epsilon_\parallel^{(1)}.$$

The result is identical with Eq. (17), where now we must replace Eq. (18) and Eq. (19) by

$$F' = \int_0^\infty dr \; r\omega_p^2 \left|\frac{d\psi}{dr}\right|^2 ; \tag{24}$$

$$G' = k_\parallel^2 \int_0^\infty dr \; r\omega_p^2 \left|\psi\right|^2 . \tag{25}$$

The formula for γ simplifies as before when we specialize considerations.

This case may be generalized without difficulty to include azimuthal dependence in ψ.

III. GROWTH RATE FROM BEAM INJECTION

We imagine that a low temperature beam of electrons moving parallel to $\vec{B}$ is superposed on the Maxwellian which was considered in Section II. If we ignore thermal motion within the beam altogether and assume it is cylindrically symmetric, the part of the electron distribution function arising from the beam has the form

$$f_b(v_{\parallel},r) = D(r)\delta\left(v_{\parallel}-v_0-\alpha(r)\right) \tag{26}$$

where $\alpha(r) + v_0$ is the velocity with which particles at a distance r from the axis of the system are moving; we assume $\alpha(0) = 0$. In general, $\alpha(r)$ will increase with r, since we may imagine the beam to have arisen as a result of shooting electrons from an electron gun into a potential profile something like that shown in Fig. 2.

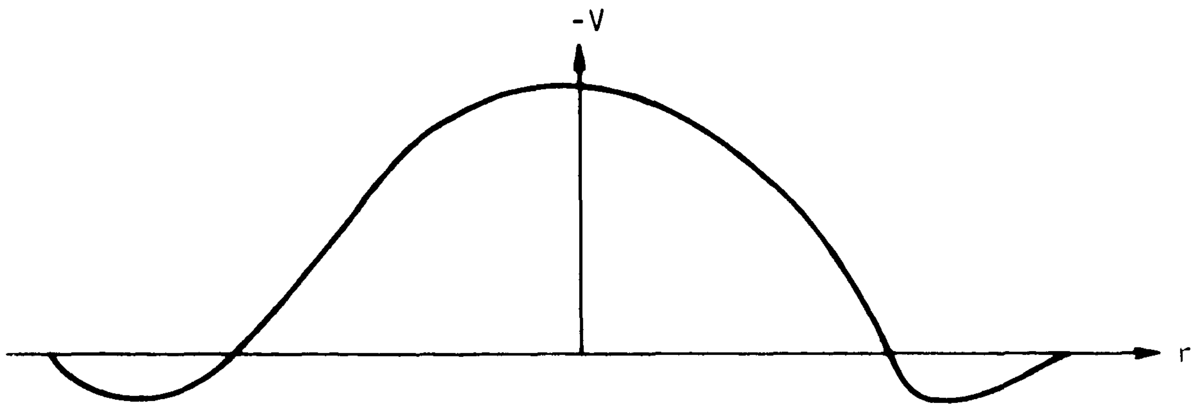

Fig. 2 -- General form of observed potential in plasma confined by magnetic field.

We are primarily interested in simple modes which propagate through the plasma as a whole, that is, in which all the electrons in a cross-sectional slice participate. This being the case, we may expect that for a beam distribution function which varies radially as does (26), the wave will see an effective distribution averaged over r. In the case of interest, the local beam density is much smaller than the density of the background Maxwellian,

$$D(r) \ll n(r) \tag{27}$$

To include the effects of the beam, we retrace the argument leading to Eq. (17). If we assume $\omega_0 \ll \Omega$, so that only the lower branch contributes to γ, the equation becomes

$$\frac{\gamma}{\omega_0} = \frac{1}{G'} \pi^{\frac{1}{2}} \frac{\omega_0^3}{(k_{\parallel} v_T)^3} e^{-\left(\frac{\omega_0}{k_{\parallel} v_T}\right)^2} G' , \tag{28}$$

where

$$G' = \int_0^{\infty} r \, dr \, \omega_p^2 |\psi|^2 .$$

Equation (28) reduces to the usual expression for linear Landau damping, since G' cancels. However, if we write (28) for a general distribution function, it takes the form

$$\frac{\gamma}{\omega_0} = -\frac{1}{G'} \int_0^{\infty} r \, dr |\psi|^2 \frac{1}{2} \omega_0^2 \, \mathrm{Im} \, \epsilon_{\parallel} , \tag{29}$$

where $\epsilon_{\parallel}$ is the usual dielectric function

$$\epsilon_{\parallel} = 1 + \frac{\omega_p^2}{k^2}\int dv_{\parallel}\, \frac{\partial f/\partial v_{\parallel}}{\omega/k - v_{\parallel}} \,.$$

When we write $f = f_m + f_{beam}$, the contribution of the beam to the growth (decay) rate is, for $\gamma \ll \omega_0$

$$\left(\frac{\gamma}{\omega_0}\right)_{beam} = \frac{1}{G'}\int_0^{\infty} r\, dr\, |\psi|^2\, \omega_p^2\, \frac{\pi}{2}\, \omega_0^2 \int_{-\infty}^{\infty} dv\, \frac{\partial g}{\partial v}\, \delta\left(v - \frac{\omega_0}{k}\right) \tag{30}$$

where

$$g = D(r)\, \delta\left(v - v_0 - \alpha(r)\right).$$

In calculating $\frac{1}{G'}$, we can ignore f_b, since the beam density is much smaller than the background density.

Equation (30) may be rewritten in order to investigate particular choices of $\alpha(r)$ and $D(r)$. To do this we assume that $\alpha(r)$ is monctone increasing, so that

$$\delta\left(v - v_0 - \alpha(r)\right) = \frac{1}{\alpha'(r_0)}\, \delta(r - r_0),$$

where $\alpha(r_0) = v - v_0$ and $\alpha' = \frac{d\alpha}{dr}$. Then integrating by parts we have

$$\left(\frac{\gamma}{\omega_0}\right)_{\text{beam}} = \frac{1}{G'}\int_0^\infty dr\; r|\psi|^2\,\omega_p^2\,\frac{\pi}{2}\left(\frac{\omega_0}{k}\right)^2\int_{-\infty}^{\infty} dv\;\delta\left(v-\frac{\omega_0}{k}\right) D(r)\,\frac{\partial}{\partial v}\left[\frac{1}{\alpha'(r_0)}\,\delta(r-r_0)\right]$$

$$= \frac{1}{G'}\,\frac{\pi}{2}\left(\frac{\omega_0}{k}\right)^2\int_0^\infty dr\; r|\psi|^2\,\omega_p^2\,D(r)\int_{-\infty}^{\infty} dv\,\frac{\partial}{\partial v}\,\delta\left(v-\frac{\omega_0}{k}\right)\frac{\delta(r-r_0)}{\alpha'(r_0)}$$

$$= -\,\frac{1}{G'}\,\frac{\pi}{2}\left(\frac{\omega_0}{k}\right)^2\int_{-\infty}^{\infty} dv\,\frac{\partial}{\partial v}\,\delta\left(v-\frac{\omega_0}{k}\right)\left[\frac{r|\psi|^2\,\omega_p^2\,D(r)\Theta(r_0)}{\alpha'(r_0)}\right]_{r=r_0}$$

$$= \frac{1}{G'}\,\frac{\pi}{2}\left(\frac{\omega_0}{k}\right)^2\frac{\partial}{\partial v}\left\{\left[\frac{r|\psi|^2\,\omega_p^2\,D(r)\Theta(r_0)}{\alpha'(r_0)}\right]_{r=r_0}\right\}_{v=\frac{\omega_0}{k}} . \tag{31}$$

We can write

$$\frac{\partial}{\partial v} = \left.\frac{\partial r}{\partial v}\right|_{r_0}\frac{\partial}{\partial r} = \frac{1}{\alpha'(r_0)}\,\frac{\partial}{\partial r} ,$$

so

$$\left(\frac{\gamma}{\omega_0}\right)_{\text{beam}} = \frac{1}{G'}\,\frac{\pi}{2}\left(\frac{\omega_0}{k}\right)^2\frac{1}{\alpha'(r_0)}\,\frac{\partial}{\partial r}\left[\frac{r|\psi|^2\,\omega_p^2\,D(r)\Theta(r)}{\alpha'(r)}\right]_{r=\alpha^{-1}\left(\frac{\omega_0}{k}-v_0\right)}$$

$$= \frac{1}{G'}\,\frac{\pi}{2}\left(\frac{\omega_0}{k}\right)^2\frac{1}{\alpha'(r_0)}\left\{\frac{r_0|\psi|^2\,\omega_p^2\,D(r_0)\delta(r_0)}{\alpha'(r_0)} + \Theta(r_0)\cdot\frac{\partial}{\partial r}\left[\frac{r|\psi|^2\,\omega_p^2\,D(r)}{\alpha'(r)}\right]\right\}_{r=\alpha^{-1}\left(\frac{\omega_0}{k}-v_0\right)=r_0} . \tag{32}$$

This shows that the contribution to γ for a particular phase velocity ω_0/k comes from particles at a distance r_0 sufficient to make

$$v_0 + \alpha(r_0) = \frac{\omega_0}{k} .$$

For $\omega_0/k < v_0$, there are no such particles, since $\alpha(r) > 0$; so if $\frac{rD(r)}{\alpha'(r)} > 0$ as $r \to 0$, there is a jump in the effective velocity distribution, yielding a δ function dependence in γ. This δ function results from our assumption that the beam had zero temperature; in fact, it is smoothed out for finite beam temperatures.

Following Fig. 2 , let us examine some plausible choices for $\alpha(r)$. We assume $D(r) = D$, a constant, and assume that ω_p^2 and $|\psi|^2$ are roughly constant out to some value $r = R$.

The first choice is

$$\alpha(r) = Ar^2, \qquad A = \text{constant.} \tag{33}$$

Then $\alpha'(r) = 2Ar$, and $r_0 = \left(\frac{v-v_0}{A}\right)^{1/2}$. We see that there is a jump in the effective distribution, since

$$\lim_{r \to 0+} \left(\frac{r}{\alpha'(r)}\right) = \frac{1}{2A} = \text{const.}$$

Formula (32) yields

$$\left(\frac{\gamma}{\omega_0}\right)_{\text{beam}} = \frac{1}{G'} \frac{\pi}{2} \left(\frac{\omega_0}{k}\right)^2 \frac{1}{2\left[A(v-v_0)\right]^{1/2}} \frac{D|\psi|^2 \omega_p^2}{2A} \left[\delta(r_0) - \delta(r_0-R)\right] . \tag{34}$$

The peak at $r_0 = R$ is of course rounded if we retain a realistic radial dependence for $\omega_p^2|\psi|^2$.

For the second choice of $\alpha(r)$, we argue as follows: an electron will have kinetic energy

$$\frac{1}{2} mv^2 = E_0 - V, \tag{35}$$

where E_0 is the "muzzle" energy of the electron gun and V is shown in Fig. 2. If V has roughly parabolic dependence or r, then

$$\begin{aligned} v &= \left[\frac{2E_0}{m} - \frac{2V}{m}\right]^{1/2} \\ &\approx \left[v_0^2 + \nu^2 r^2\right]^{1/2} \end{aligned} \tag{36}$$

Thus

$$\alpha(r) = (v_0^2 + \nu^2 r^2)^{1/2} - v_0;$$

$$\frac{1}{\alpha'(r)} \frac{d}{dr} \left(\frac{r}{\alpha'(r)}\right) = \frac{1}{\nu^2} .$$

Now formula (37) tells us that for $0 < r_0 < R$,

$$\left(\frac{\gamma}{\omega_0}\right)_{\text{beam}} = \frac{1}{G'} \frac{\pi}{2} \left(\frac{\omega_0}{k}\right)^2 D\, \omega_p^2 \,|\psi|^2 \frac{1}{\nu^2} .$$

as shown in Fig. 3.

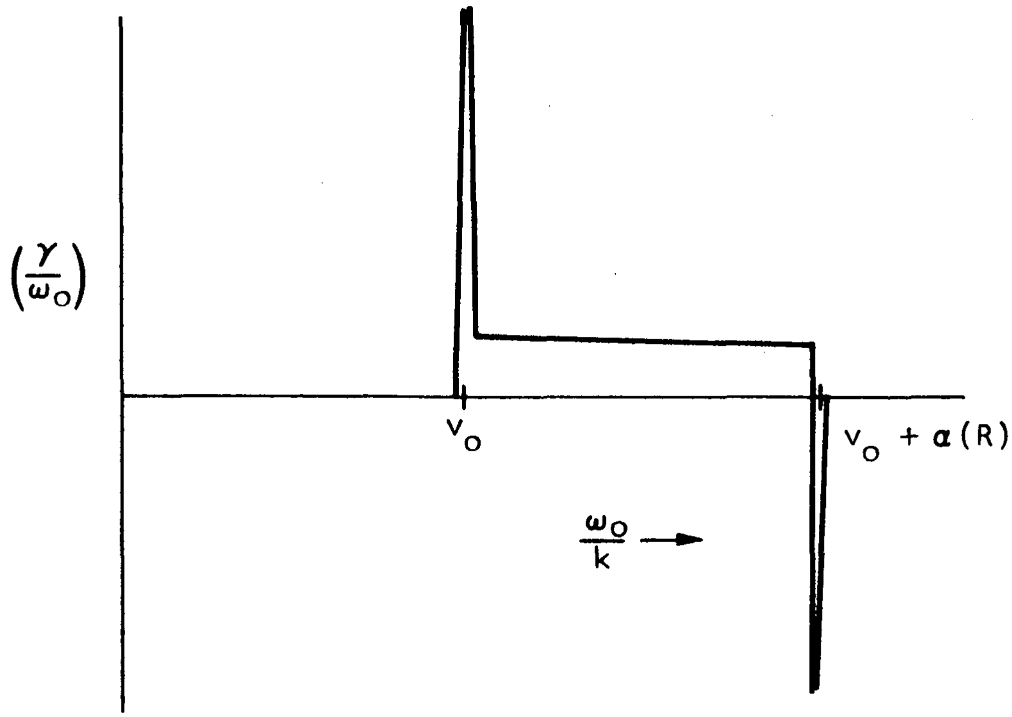

Fig. 3 -- Growth rate resulting from beam with dependence $\alpha(r) = v_0^2 + \nu^2 r^2)^{\frac{1}{2}} - v_0$.

It is not necessary to treat finite beam temperature to remove the sharp peak in γ. Thus far we have taken $D(r)$ to be a constant. If a template is placed before the stream of electrons from the electron gun, it can screen some out, so that $D(r)$ may vanish at $r = 0$. For example, let the template be in the form of a screen whose edges have roughly the shape shown in Fig. 4, satisfying the equation $r^2 = a^2 \sin\theta$.

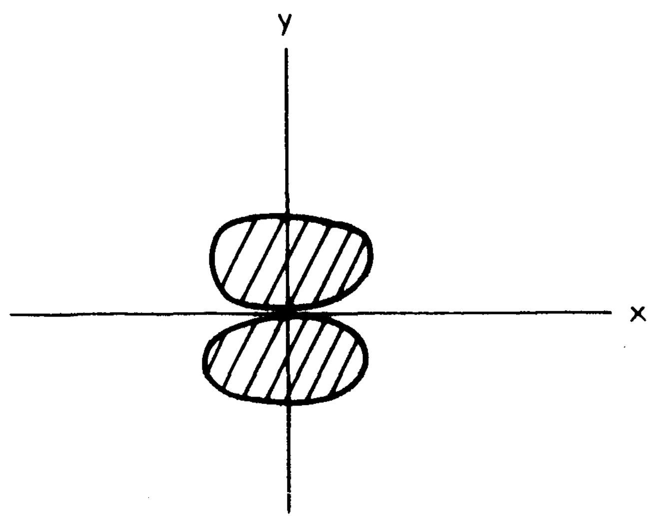

Fig. 4 -- Outline of a template with shape described by $r^2 = a^2 \sin\theta$.

Then $D(r) \sim 4 \int_0^{\sin^{-1} \frac{r^2}{a^2}} d\Theta \approx 4 \frac{r^2}{a^2}$ for r small.

Now, taking $\omega_p^2 |\psi|^2$ gradually decreasing as r increases (instead of a sudden drop at $r = R$), we have for γ a curve of the form shown in Fig. 5.

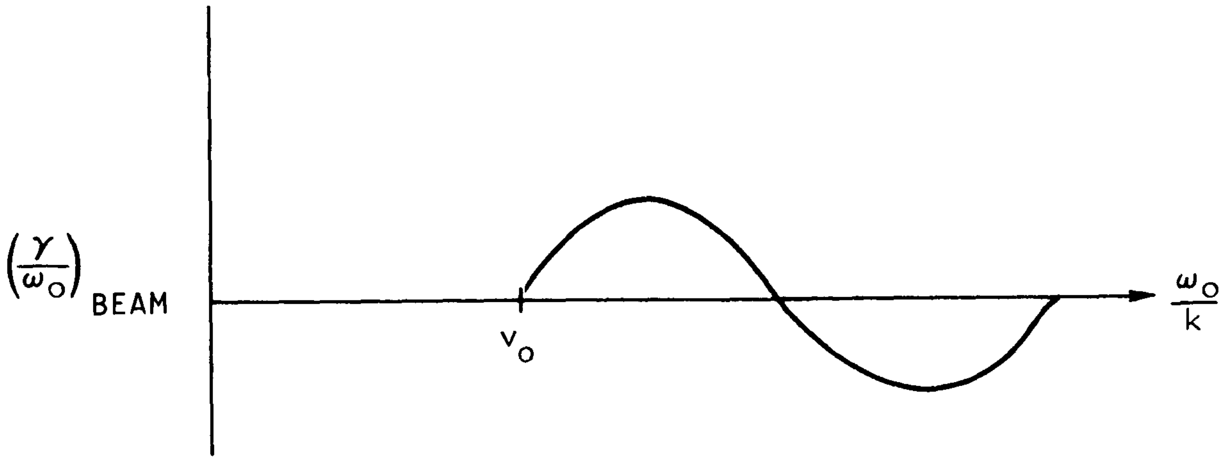

Fig. 5 -- γ/ω_0 modified by template and radial dependence of plasma.

Note that the area under the curve sums to zero. This follows from the next to last line of Eq. (31), which expresses γ as a total derivative with respect to v of a quantity which vanishes as $v \to v_0$ and $v \to \infty$.

We conclude that it is possible to produce the "gentle bump" instability discussed in linear and quasilinear[6] analyses of infinite plasmas in a bounded laboratory plasma by proper adjustment of the injected electron beam.

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